

Functional Hecke algebras and simple Bernstein blocks of a p -adic GL_n in non-defining characteristic

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Abstract

Let $G_n = \mathrm{GL}_n(F)$, where F is a non-archimedean local field with residue characteristic p and where $n = 2k$ is even. In this article, we investigate a question occurring in the decomposition of the category of ℓ -modular smooth representations of G_n into Bernstein blocks (where $\ell \neq p$). The easiest block not investigated in [7] is the one defined by the standard parabolic subgroup with Levi factor $M = \mathrm{GL}_k(F) \times \mathrm{GL}_k(F)$ and by an M -representation of the form $\pi_0 \boxtimes \pi_0$ with π_0 a supercuspidal $\mathrm{GL}_k(F)$ -representation. This block is Morita equivalent to a Hecke algebra which we can describe as a twisted tensor product of a finite Hecke algebra (i. e. a Hecke algebra occurring in the representation theory of the finite group $\mathrm{GL}_k(p^\alpha)$ in non-defining characteristic ℓ) and the group ring of \mathbb{Z}^2 . This enables us to describe how a conjectured connection between finite Hecke algebras (which is similar to a connection postulated by Broué in [1]) would lead to an equivalence between the described block and the unipotent block of $\mathrm{GL}_2(F^k)$, where F^k is the unramified extension of degree k over F .

1 Introduction

Let F be a non-archimedean local field with ring of integers \mathcal{O}_F , uniformizer ϖ_F and finite residue field $k_F \cong \mathbb{F}_q = \mathbb{F}_{p^\alpha}$ for some prime p . Moreover, let R be an algebraically closed field of positive characteristic $\ell \neq p$ such that R arises as residue field in some ℓ -modular system (R, \mathcal{O}_K, K) . Continuing the exposition in [7], we consider the category $\mathfrak{R}_R(G)$ of smooth R -valued representations of the group $G = G_n = \mathrm{GL}_n(F)$. In particular, recall (e.g. from Theorem 2.22 of [7]) that the level-0 part of $\mathfrak{R}_R(G)$ decomposes as a direct sum of Bernstein-blocks, each of the form

$$\mathfrak{R}_R^{[M, \pi]_G}(G) = \left\{ V \in \mathfrak{R}_R(G) \left| \begin{array}{l} \text{any irreducible JH-constituent of } V \text{ is a JH-constituent} \\ \text{of } i_{N \subset P}^G(\sigma) \text{ for some } (N, \sigma) \in [M, \pi]_G \end{array} \right. \right\},$$

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where $[M, \pi]_G$ denotes a G -equivalence class of supercuspidal pairs and $i_{N \subset P}^G$ denotes parabolic induction. So, in particular, each M is a Levi subgroup of G and π is an irreducible supercuspidal M -representation. In this paper, we will study simple Bernstein-blocks of the smallest non-trivial form, i. e. where $M \cong \mathrm{GL}_k(F) \times \mathrm{GL}_k(F)$ and $\pi \cong \pi_0 \boxtimes \pi_0$, where $n = 2k$ and π_0 is an irreducible supercuspidal level-0 $\mathrm{GL}_k(F)$ -representation¹.

Now, if $\mathcal{V} \in \mathfrak{R}_R^{[M, \pi]G}(G)$ is a pro-generator, we get an equivalence

$$\mathfrak{R}_R^{[M, \pi]G}(G) \underset{\text{Morita}}{\cong} \mathrm{End}_G(\mathcal{V}).$$

As π_0 is level-0, it is of the form $\mathrm{ind}_{Z \mathrm{GL}_k(\mathcal{O}_F) \times Z \mathrm{GL}_k(\mathcal{O}_F)}^M(\rho)$, where $\rho = \rho_0 \boxtimes \rho_0$ and ρ_0 it is inflated from an irreducible supercuspidal representation of the finite group $\mathcal{G}_k = \mathrm{GL}_k(q)$ and where Z denotes the center of $\mathrm{GL}_k(F)$. We will not distinguish between ρ_0 (resp. ρ) as a representation of $\mathrm{GL}_k(\mathcal{O}_F)$ and of \mathcal{G}_k (resp. of $\mathrm{GL}_k(\mathcal{O}_F) \times \mathrm{GL}_k(\mathcal{O}_F)$ and of $\mathcal{G}_k \times \mathcal{G}_k$).

Example 1.1. Assume that ℓ does not divide the order of \mathcal{G}_n . Then, in particular, ℓ does not divide $q^k - 1$, and this implies (see III.2.9 of [9]) that ρ_0 is projective. Along the lines of the proof of Theorem 4.3 in [7], one can show that $i_{\mathcal{P}}^G(\rho)$ is a pro-generator, where $i_{\mathcal{P}}^G$ denotes the parahoric induction² from the standard parahoric subgroup $\mathcal{P} \subset G$ uniquely characterized by $\mathcal{P}/\mathcal{P}(1) \cong \mathcal{G}_k \times \mathcal{G}_k$. The important point is to use that $R[\mathcal{G}_n] - \mathbf{Mod}$ is semisimple, so $\mathrm{Hom}_{\mathcal{G}_n}(i_{\mathcal{G}_k \times \mathcal{G}_k}^{\mathcal{G}_n}(\rho), X)$ is non-zero for any subquotient (hence, subrepresentation) X of $i_{\mathcal{G}_k \times \mathcal{G}_k}^{\mathcal{G}_n}(\rho)$. This implies

$$\mathfrak{R}_R^{[M, \pi]G}(G) \underset{\text{Morita}}{\cong} \mathcal{H}_R(G, \mathcal{P}, \rho),$$

where the object on the right hand side is the Hecke algebra of the type (\mathcal{P}, ρ) . By [9], Proposition III.3.6, one gets an isomorphism

$$\mathcal{H}_R(G, \mathcal{P}, \rho) \cong H_R(2, q^k) \cong \mathcal{H}_R(\mathrm{GL}_2(F^k), \mathcal{I}),$$

where F^k is the unramified extension of F of degree k and \mathcal{I} is the Iwahori subgroup of $\mathrm{GL}_2(F^k)$. Hence, abbreviating $G' = \mathrm{GL}_2(F^k)$ and denoting by $T' \subset G'$ the standard torus, we get an equivalence of categories

$$\mathfrak{R}_R^{[M, \pi]G}(G) \cong \mathfrak{R}_R^{[T', 1]_{G'}}(G'). \quad (1)$$

The block on the right is the unipotent block of G' . It contains the trivial representation.

The assumptions of the above example can be called banal for our context. Our aim in this article is to investigate the non-banal situation. To this end, we will say in Section 2 how one can write down a pro-generator \mathcal{V} of the block $\mathfrak{R}_R^{[M, \pi]G}(G)$, depending only on the finite-group data \mathcal{G}_k and ρ_0 . In Section 3 we will describe the multiplicative structure of “functional” Hecke algebras, i. e. algebras of the form

$$\mathcal{H}_R(G, \mathcal{P}, V) = \mathrm{End}_G(i_{\mathcal{P}}^G(V))$$

where V is a cuspidal (but not necessarily irreducible) representation of $\mathcal{M} = \mathcal{G}_k \times \mathcal{G}_k$. This will be used in Section 4 to establish the main result of this article (Corollary 4.16), which gives a decomposition

¹As [7] explained how to reduce from arbitrary blocks to simple blocks, this is the logically next case to consider, from the point of view of [7].

²For notations and definition, see chapter 2.2 in [7].

of the Hecke algebra as a twisted³ tensor product

$$\mathcal{H}_R(G, \mathcal{P}, V) \cong R[\mathbb{Z}] \otimes_R^\zeta \mathcal{H}_R(\mathcal{G}_n, \mathcal{M}, V).$$

If we apply this to the constructed pro-generator \mathcal{V} (which is of the form $i_{\mathcal{P}}^G(V)$ for a suitable \mathcal{M} -representation (ρ, V)), this gives a strong relation between the block $\mathfrak{R}_R^{[M, \pi]G}(G)$ and the block

$$\mathfrak{R}_R^{[\mathcal{M}, \rho]_{\mathcal{G}_n}}(\mathcal{G}_n) \subset \text{Rep}_R(\mathcal{G}_n).$$

This second block is Morita equivalent to $\mathcal{H}_R(\mathcal{G}_n, \mathcal{M}, V)$ and consists of those representations whose Jordan-Hölder quotients are isomorphic to Jordan-Hölder quotients of the Harish-Chandra induced of V from \mathcal{M} to \mathcal{G}_n . This decomposition also suggests a way to generalize the equivalence (1) (which we will formulate as Conjecture 5.1): If we can establish an isomorphism of the “finite” Hecke algebras

$$\mathcal{H}_R(\mathcal{G}_n, \mathcal{M}, V) \cong \mathcal{H}_R(\text{GL}_2(q^k), \mathcal{B}, V_0) \quad (2)$$

(where \mathcal{B} denotes the Borel subgroup and V_0 denotes some pro-generator of the unipotent block of $\text{GL}_2(q^k)$) and if this isomorphism is compatible with the respective twisting parameters ζ , then we get an isomorphism

$$\mathcal{H}_R(G, \mathcal{P}, V) \cong \mathcal{H}_R(\text{GL}_2(F^m), \mathcal{I}, V_0)$$

and, consequently, an equivalence of $\mathfrak{R}_R^{[M, \pi]G}(G)$ with the unipotent block of $\text{GL}_2(F^k)$. The precise requirements on the isomorphism (2) are summarized in Conjecture 5.3.

It seems plausible that a similar decomposition of the Hecke algebra as a twisted tensor product can be established with some care in more general situations (i. e. for more complicated blocks). However, the trick used in the proof of Theorem 2.2 does not carry over. So, even with a tensor decomposition theorem and a suitable version of Conjecture 5.3 at hand, the presented approach will not directly imply an isomorphism of blocks à la (1). This is the reason why we decided to restrict our attention to blocks of the mentioned type.

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2 Preliminaries on the pro-generator

Retain the notations of the previous chapter, in particular

- $G = G_n = \text{GL}_n(F)$, $\mathcal{G} = \mathcal{G}_n = \text{GL}_n(q)$ with $n = 2k$;
- $\mathcal{P} \subset G$ the standard parahoric subgroup with reductive quotient $\mathcal{P}/\mathcal{P}(1) \cong \mathcal{M} := \mathcal{G}_k \times \mathcal{G}_k$;
- ρ denotes an \mathcal{M} -representation of the form $\rho_0 \otimes \rho_0$, where ρ_0 is an irreducible supercuspidal \mathcal{G}_k -representation.

³This construction is explained in section 4.1. We would also like to point the reader to the appendix of [6], where M.-F. Vignéras gives a similar (but not identical) decomposition in a special case.

Now, inside $R[\mathcal{M}]$ -**Mod** fix a projective cover P of ρ . Define $V = P \oplus P^*$, where P^* is the contragredient representation of P . As usual, we do not distinguish between V and its inflation along $\mathcal{P}(1)$ to a representation of \mathcal{P} .

Proposition 2.1. *V is cuspidal (in the sense that it vanishes under non-trivial Harish-Chandra functors).*

Proof. The statement is certainly true for P in place of V , as the projective cover is formed in the (supercuspidal) block associated to ρ , so all subquotients of P are isomorphic to ρ , i. e. are cuspidal. That this property is not disturbed by adding the contragredient follows from [2], 2.2e Lemma. \square

Let $[M, \pi]$ be the supercuspidal pair associated to the type (\mathcal{P}, ρ) (cf. [7], 2.2).

Theorem 2.2. *The parahorically induced $\mathcal{V} = \text{ind}_{\mathcal{P}}^G(V)$ is a pro-generator of the block $\mathfrak{R}^{[M, \pi]_G}(G)$.*

Proof. It is clear that \mathcal{V} is finitely generated and projective, as these properties are respected by parahoric induction (and as by [9], the projective cover is finite-length, hence finitely generated). Now, by Frobenius reciprocity and transitivity of parahoric induction, we have

$$\text{Hom}_G(\mathcal{V}, W) = \text{Hom}_{\mathcal{G}}(i_{\mathcal{M}}^{\mathcal{G}}(V), W^{\mathcal{K}})$$

for any non-zero $W \in \mathfrak{R}^{[M, \pi]}(G)$, where $\mathcal{K} = \text{GL}_n(\mathcal{O}_F)$ denotes the maximal compact subgroup and we can assume that W is irreducible. As W is level-0, we have $W^{\mathcal{K}} \neq 0$. Thus, in order to show that the above Hom-set does not vanish, it is sufficient (by [7], 2.16) to show that there exists a non-zero map

$$i_{\mathcal{M}}^{\mathcal{G}}(V) \rightarrow X$$

for any subquotient X of $i_{\mathcal{M}}^{\mathcal{G}}(\rho)$. By ([2], 2.4a), this representation has precisely two subquotients, each one occurring with multiplicity 1. So we can write

$$0 \rightarrow Y \rightarrow i_{\mathcal{M}}^{\mathcal{G}}(\rho) \rightarrow Z \rightarrow 0.$$

Now, as $P \twoheadrightarrow \rho$ and Harish-Chandra induction is exact and respects direct sums, we get a surjection $i_{\mathcal{M}}^{\mathcal{G}}(V) \twoheadrightarrow \rho$. Composing this with the projection from the short exact sequence, we get a non-zero map if $X = Z$.

For $X = Y$, we use that taking the contragredient is an exact contravariant functor, so there is a projection $i_{\mathcal{M}}^{\mathcal{G}}((\rho)^*) \cong i_{\mathcal{M}}^{\mathcal{G}}(\rho)^* \twoheadrightarrow Y$ (the isomorphism is [2], 2.2e; note that we have used here that an irreducible representation is isomorphic to its contragredient, as mentioned in the remarks before Lemma 2.2e in [2]). By [9], III.2.9 Theoreme 2.b, all irreducible constituents of P are of the form ρ , i. e. $\rho \hookrightarrow P$. But this implies the existence of a surjection $P^* \twoheadrightarrow (\rho)^*$. This can again be induced and composed with the map above to finish off the case $X = Y$. [2]. \square

3 Functional Hecke-algebras

We use the notation from above, except that we allow (V, σ) to be any finite-dimensional \mathcal{M} -representation which we assume to be cuspidal (in the sense that all non-trivial Harish-Chandra functors vanish on

it) but not necessarily irreducible. We will not distinguish between V and its inflation along $\mathcal{P}(1)$ to a representation of \mathcal{P} . Our aim is to study the R -algebra

$$\mathcal{H}_R(G, \mathcal{P}, V) = \text{End}_G(\text{ind}_{\mathcal{P}}^G(V)).$$

Following [9], I.8.5, this algebra can be characterized as the convolution algebra of functions

$$\varphi : G \rightarrow \text{End}_R(V)$$

supported on a finite number of double cosets $\mathcal{P}g\mathcal{P}$ such that

$$\varphi(p_1gp_2) = \sigma(p_1)\varphi(g)\sigma(p_2) \quad \text{for all } g \in G, p_1, p_2 \in \mathcal{P}. \quad (3)$$

If $\mathcal{H}_R(\mathcal{P}g\mathcal{P}, \mathcal{P}, V) \subset \mathcal{H}_R(G, \mathcal{P}, V)$ denotes the subspace of functions supported on the single coset $\mathcal{P}g\mathcal{P}$, we have a vector space decomposition

$$\mathcal{H}_R(G, \mathcal{P}, V) = \bigoplus_{\eta \in \tilde{\mathcal{W}}} \mathcal{H}_R(\mathcal{P}\eta\mathcal{P}, \mathcal{P}, V),$$

where $\tilde{\mathcal{W}}$ denotes a set of representatives for $\mathcal{P} \backslash G / \mathcal{P}$. With $\varpi = \varpi_F$, let

$$\mathcal{W} = \left\{ \text{monomial } 2k \times 2k\text{-matrices with entries in } \varpi^{\mathbb{Z}} \right\}$$

be the affine Weyl group of G . Then, according to e. g. [11], p. 43, we can take as $\tilde{\mathcal{W}}$ a set of representatives of $Y \backslash \mathcal{W} / Y$, where $Y \cong S_k \times S_k$ denotes the Young subgroup associated to the partition (k, k) of $n = 2k$. This means we can take

$$\tilde{\mathcal{W}} = \left\{ v \cdot \text{diag}(\varpi^{a_1}, \dots, \varpi^{a_{2k}}) \mid a_i \in \mathbb{Z}, v \in \mathbb{W} \right\},$$

where \mathbb{W} is a fixed set of representatives for $Y \backslash S_n / Y$, which we assume for convenience to include $1 = \text{diag}(1, \dots, 1)$ and $w = (1, k+1)(2, k+2) \dots (k, 2k)$, or, in matrix notation,

$$w = \begin{pmatrix} & & & 1 \\ & & \ddots & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & & 1 \end{pmatrix}.$$

Denote

$$W = \left\{ v \cdot \text{diag}(\varpi^{a_1}, \dots, \varpi^{a_{2k}}) \in \tilde{\mathcal{W}} \mid a_1 = a_2 = \dots = a_k, a_{k+1} = a_{k+2} = \dots = a_{2k}, v \in \{1, w\} \right\}$$

and remark that this forms a subgroup of \mathcal{W} isomorphic to the affine Weyl group of $\text{GL}_2(F)$.

Lemma 3.1. *The support of $\mathcal{H}_R(G, \mathcal{P}, V)$ is contained in $\mathcal{P}W\mathcal{P}$, i. e.*

$$\mathcal{H}_R(\mathcal{P}\eta\mathcal{P}, \mathcal{P}, V) \neq 0 \text{ and } \eta \in \tilde{\mathcal{W}} \Rightarrow \eta \in W.$$

Proof. This is Lemma 1.1 (p. 15) in [8]. (The representation there is irreducible, but the argument only needs cuspidality.) \square

Now, if $\eta = v \cdot \text{diag}(\varpi^{a_1}, \dots, \varpi^{a_{2k}})$, write $\delta = a_1 - a_{k+1}$. Moreover, abbreviate $\mathcal{O} = \mathcal{O}_F$ and $\mathfrak{P} = \varpi \cdot \mathcal{O}$.

Lemma 3.2. Denote $\mathcal{P}^{(\eta)} = \mathcal{P} \cap \eta \mathcal{P} \eta^{-1}$, then we have

- $$\mathcal{P}^{(\eta)} = \begin{pmatrix} \text{GL}_k(\mathcal{O}) & \mathbb{M}_{k \times k}(\varpi^\delta \mathcal{O}) \\ \mathbb{M}_{k \times k}(\mathfrak{P}) & \text{GL}_k(\mathcal{O}) \end{pmatrix} \quad \text{if } v = 1 \text{ and } \delta \geq 0;$$
- $$\mathcal{P}^{(\eta)} = \begin{pmatrix} \text{GL}_k(\mathcal{O}) & \mathbb{M}_{k \times k}(\mathcal{O}) \\ \mathbb{M}_{k \times k}(\varpi^{1-\delta} \mathfrak{P}) & \text{GL}_k(\mathcal{O}) \end{pmatrix} \quad \text{if } v = 1 \text{ and } \delta < 0;$$
- $$\mathcal{P}^{(\eta)} = \begin{pmatrix} \text{GL}_k(\mathcal{O}) & \mathbb{M}_{k \times k}(\varpi^{1-\delta} \mathcal{O}) \\ \mathbb{M}_{k \times k}(\mathfrak{P}) & \text{GL}_k(\mathcal{O}) \end{pmatrix} \quad \text{if } v = w \text{ and } \delta \leq 0;$$
- $$\mathcal{P}^{(\eta)} = \begin{pmatrix} \text{GL}_k(\mathcal{O}) & \mathbb{M}_{k \times k}(\mathcal{O}) \\ \mathbb{M}_{k \times k}(\mathfrak{P}^\delta) & \text{GL}_k(\mathcal{O}) \end{pmatrix} \quad \text{if } v = w \text{ and } \delta > 0.$$

Proof. Straight-forward matrix calculation. □

We easily get an identification

$$\begin{aligned} \mathcal{H}_R(\mathcal{P} \eta \mathcal{P}, \mathcal{P}, V) &= \left\{ \psi : V \rightarrow V \mid \psi \circ \sigma(x) = \sigma(\eta x \eta^{-1}) \circ \psi \text{ for all } x \in \mathcal{P}^{(\eta^{-1})} \right\} \\ &\cong \begin{cases} I_1 = \text{End}_{\mathcal{M}}(V) & \text{if } \eta \text{ diagonal;} \\ I_w = \text{Hom}_{\mathcal{M}}(V^w, V) \cong \text{Hom}_{\mathcal{M}}(V, V^w) & \text{otherwise;} \end{cases} \end{aligned}$$

where V^w denotes the w -conjugate representation of V , i. e. we let $m \in \mathcal{M}$ act on $v \in V$ as $(w m w^{-1}) \cdot v$. We will consequently use the notation $[\eta]_f$ for the element of $\mathcal{H}_R(P \eta P, P, V)$ with $\varphi(\eta) = \psi = f$, where f is an element of I_1 or I_w depending on whether η is diagonal or not.

Fix the following notation, which is clearly motivated by the isomorphism of W with the affine Weyl group of $\text{GL}_2(F)$:

- $t = \begin{pmatrix} & & & 1 \\ & & & \ddots \\ \varpi & & & \\ & \ddots & & \\ & & \varpi & \\ & & & 1 \end{pmatrix};$
- $w' = t w t^{-1};$
- $W_0 = \{x \in W \mid \det(x) = \pm 1\}.$

Using this isomorphism, W is an affine Coxeter group with length function l satisfying $l(w) = l(w') = 1, l(t) = 0$. On the other hand, \mathcal{W} is also an affine Coxeter group equipped with a length function l' .

Proposition 3.3. $l'|W$ is a constant multiple of l .

Proof. This is (5.6.14) (p. 192) of [4]. Notice that there it is written that $l'|W = n \cdot l$, but this is not correct and it should be $l'|W = n^2 \cdot l$. □

Of fundamental importance is the operator

$$T^* = \sum_{g \in \mathrm{GL}_k(q)} \begin{pmatrix} g & \\ & -g^{-1} \end{pmatrix} \in R[\mathcal{M}] \subset R[\mathcal{G}],$$

which already appears in [8], p. 10, (5.13). T^* lies in I_w but generally not in I_1 , as a quick calculation reveals.

Proposition 3.4. (i) *If ℓ divides $q - 1$, then $(T^*)^2 = 0$.*

(ii) *If f is in I_1 (resp. I_w), then $T^*f = fT^*$ in I_w (resp. I_1).*

Proof. (ii) is obvious, so we only treat (i). First, we have

$$(T^*)^2 = \sum_{g \in \mathrm{GL}_k(q)} \sum_{h \in \mathrm{GL}_k(q)} \begin{pmatrix} gh & \\ & g^{-1}h^{-1} \end{pmatrix} = \sum_{g \in \mathrm{GL}_k(q)} \sum_{h \in \mathrm{GL}_k(q)} \begin{pmatrix} g & \\ & hg^{-1}h^{-1} \end{pmatrix} = \sum_{g \in \mathrm{GL}_k(q)} \sum_{h \in C(g)} (\#Z_h) \begin{pmatrix} g & \\ & h \end{pmatrix},$$

where $C(g) = \{hgh^{-1} \mid h \in \mathrm{GL}_k(q)\}$ and $Z_h = \{x \in \mathrm{GL}_k(q) \mid xhx^{-1} = h\}$. As $k_F^\times \cdot \mathbf{1}$ is a subgroup of Z_h for any $h \in G$, ℓ divides all the coefficients $\#Z_h$ in the above expression. \square

It is easy to see that the first condition in the Proposition is not true in general. In fact, one can produce examples (i. e. list combinations of ℓ, q, n) where T^* is not even nilpotent.

Definition 3.5. (i) Denote by $\tau = \tau_{\ell, q, k} := q^{(k^2)} \bmod \ell$. It follows by our assumption $(\ell, q) = 1$ that τ is non-zero, so $\tau \in \{1, \dots, \ell - 1\}$.

(ii) If f is in I_1 or I_w , $\eta \in W$ and $j \in \mathbb{N}$, we write $[\eta]_f^j$ for $[\eta]_{(T^*)^j f}$ and $[\eta]^j$ for $[\eta]_1^j$. Observe that this makes sense only for even j (in the case that $f \in I_1$ and η diagonal or $f \in I_w$ and η non-diagonal) or for odd j (in the case that $f \in I_w$ and η diagonal or $f \in I_1$ and η non-diagonal). Clearly, $[\eta]_f = [\eta]_f^0$.

Our aim is to prove the following:

Theorem 3.6. *Let $[\eta]_f, [\delta]_g \in \mathcal{H}_R(G, \mathcal{P}, V)$ and, in the case $l(\eta) > 1$, write $\eta = \eta'v$ with $v \in S = \{w, w'\}$. Then*

$$[\eta]_f * [\delta]_g = \begin{cases} [\eta\delta]_{fg} & \text{if } l(\eta\delta) = l(\eta) + l(\delta); \\ \tau \cdot [\eta\delta]_{fg} + [\eta'\delta]_{fg}^1 & \text{if } l(\eta) = l(\delta) = 1 \text{ and } l(\eta\delta) = 0. \end{cases}$$

(The theorem does not make a statement for the case $l(\eta\delta) \neq l(\eta) + l(\delta)$ and $\max(l(\eta), l(\delta)) > 1$.)

Corollary 3.7. *The above theorem completely determines the multiplication in $\mathcal{H}_R(G, \mathcal{P}, V)$.*

Proof of the corollary. We show how the theorem can be used inductively to evaluate the product $[\eta]_f * [\delta]_g$, where the induction is on the number $l_{\eta, \delta} = l(\eta) + l(\delta)$. The above theorem allows evaluation if $l_{\eta, \delta} \leq 2$. Moreover, the above theorem establishes

$$\text{Fact A : } \quad l(\eta\delta) < l_{\eta, \delta} \Rightarrow [\eta]_f * [\delta]_g = \sum_I [\epsilon_i]_{h_i} \text{ with } l(\epsilon_i) < l_{\eta, \delta} \text{ for all } i \in I$$

under the condition that $l_{\eta,\delta} \leq 2$.

For the induction step, assume that Fact A and multiplication of $[\eta]_f$ and $[\delta]_g$ are established as long as $l_{\eta,\delta} < m$ for some m . We explain how to prove Fact A and how to evaluate $[\eta]_f * [\delta]_g$ if $l_{\eta,\delta} = m$:

If $l(\eta\delta) = m$, we can directly apply the above theorem, so assume inequality. Moreover, assume for the moment that $l(\eta) \geq 2$. This means, we can break up $\eta = \eta_1\eta_2$ in a way that $l(\eta_1) + l(\eta_2) = l_{\eta_1,\eta_2}$, both $l(\eta_1)$ and $l(\eta_2)$ are nonzero, and η_1 is diagonal. Explicitly, if $\eta = t^\alpha w_1 \dots w_a$ is a minimal expression, we can take

$$(\eta_1)(\eta_2) = \begin{cases} (t^\alpha w_1)(w_2 \dots w_a) & \text{if } \alpha \text{ is odd;} \\ (t^{\alpha-1} w'_1)(tw_2 \dots w_a) & \text{if } \alpha > 0 \text{ is even;} \\ (t^{-1} w'_1)(tw_2 \dots w_a) & \text{if } \alpha = 0, w_1 = w; \\ (tw'_1)(t^{-1} w_2 \dots w_a) & \text{if } \alpha = 0, w_1 = w'. \end{cases}$$

Here, the operator $w_1 \mapsto w'_1$ is given by interchanging the symbols “ w ” and “ w' ”.

In this situation, the inequality $l(\eta\delta) < m$ implies $l(\eta_2\delta) < l_{\eta_2,\delta} = l_{\eta,\delta} - l(\eta_1) < m$ (as can be easily seen from writing $\delta = w_{a+1} \dots w_{a+b} t^\beta$ and noting $l(\eta\delta) < m \Leftrightarrow w_a = w_{a+1}$). So, in

$$[\eta]_f * [\delta]_g = [\eta_1]_1 * ([\eta_2]_f * [\delta]_g),$$

the expression in the brackets can be calculated because $l_{\eta_2,\delta} < m$:

$$[\eta]_f * [\delta]_g = [\eta_1]_1 * \sum_I [\epsilon_i]_{h_i},$$

where $l(\epsilon_i) < l_{\eta_2,\delta} = m - l(\eta_1)$ (because of Fact A applied to $[\eta_2]_f * [\delta]_g$). Therefore, each summand $[\eta_1]_1 * [\epsilon_i]_{h_i}$ can be calculated. Moreover, if

$$[\eta_1]_1 * [\epsilon_i]_{h_i} = \sum_{j \in J_i} [\nu_j^i]_{h_j^i},$$

we can use the inequality $l_{\eta_1,\epsilon_i} < m - l(\eta_1) + l(\eta_1) = m$ to apply either Fact A (if $l_{\eta_1,\epsilon_i} < l(\eta_1) + l(\epsilon_i)$) or the above theorem (if $l_{\eta_1,\epsilon_i} = l(\eta_1) + l(\epsilon_i)$), giving $l(\nu_j^i) < m$ and establishing Fact A for $[\eta]_f * [\delta]_g$.

If the initial assumption on η is not met, we can use an analogous argument from the right (i. e. breaking up $\delta = \delta_2\delta_1$). \square

3.1 Proof of Theorem 3.6

Our main tool is the following formula:

Proposition 3.8. *Let $[\eta]_f, [\delta]_g \in \mathcal{H}_R(G, \mathcal{P}, V)$. Then*

$$[\eta]_f * [\delta]_g = \sum_{\epsilon \in W \cap \mathcal{P} \eta \mathcal{P} \delta \mathcal{P}} [\epsilon]_{h_\epsilon},$$

where

$$h_\epsilon = \sum_{\substack{(k_1, k_2) \in (\mathcal{P}^{(\eta^{-1})} \setminus \mathcal{P}) \times (\mathcal{P}^{(\delta^{-1})} \setminus \mathcal{P}) \\ \text{such that } k_0 := \epsilon k_2^{-1} \delta^{-1} k_1^{-1} \eta^{-1} \in \mathcal{P}}} \rho(k_0) \circ f \circ \rho(k_1) \circ g \circ \rho(k_2). \quad (4)$$

For notational simplicity, we will refer to the pairs (k_1, k_2) which fulfill the condition described under the summation sign as “admissible” pairs (with respect to the problem of evaluating h_ϵ).

Proof. This is a slightly reformulated version of the formula given in [10], II.2. \square

It is important to note that the indices under both summation signs do not depend on R, V, f and g . This means, we can get a lot of information just from looking at the spherical integral Hecke algebra $\mathcal{H}_{\mathbb{Z}}(G, \mathcal{P}, 1)$.

Proposition 3.9. $\#(\mathcal{P}x\mathcal{P}/\mathcal{P}) = q^{l'(x)}$ for $x \in W$.

Proof. By [4], Chapter 5.5, there is a subgroup $\mathfrak{M}^\times \subset \mathcal{P}$ such that $\mathcal{P} = \mathfrak{M}^\times \mathcal{I}$ (with \mathcal{I} denoting the Iwahori subgroup) and W normalizes \mathfrak{M}^\times . (In our case, \mathfrak{M}^\times is $\mathrm{GL}_k(\mathcal{O}) \times \mathrm{GL}_k(\mathcal{O})$ understood as block matrices.) So we can use the solid bijections in

$$\begin{array}{ccccc} \mathcal{P}x\mathcal{P}/\mathcal{P} & \xlongequal{\quad} & \mathcal{I}x\mathcal{P}/\mathcal{P} & \xleftarrow{\quad\quad\quad} & \mathcal{I}x\mathcal{I}/\mathcal{I} \\ & & \updownarrow & & \updownarrow \\ & & \mathcal{I}/(\mathcal{I} \cap x\mathcal{P}x^{-1}) & \longleftrightarrow & \mathcal{I}/(\mathcal{I} \cap x\mathcal{I}x^{-1}) \end{array}$$

to establish that the dotted arrow is a bijection. The nature of the vertical arrows is clear. The bottom horizontal arrow is a bijection by use of the identity

$$\mathcal{I} \cap x\mathcal{P}x^{-1} = \mathcal{I} \cap x\mathcal{I}x^{-1},$$

which follows from a straightforward matrix computation, invoking the block structure of x (i. e. the fact that $x \in W$). Finally, the cardinality of $\mathcal{I}x\mathcal{I}/\mathcal{I}$ is $q^{l'(x)}$ e. g. by [4], 5.4.3.(ii). \square

Theorem 3.10. If $l(\eta\delta) = l(\eta) + l(\delta)$ and $\eta, \delta \in W$, then $[\eta] * [\delta] = [\eta\delta]$ in $\mathcal{H}_{\mathbb{Z}}(G, \mathcal{P}, 1)$.

Proof. Using that l and l' differ by a non-zero factor, we can use Lemma 5.6.12 of [4] to deduce from the addition of lengths that $\mathcal{P}\eta\mathcal{P}\delta\mathcal{P} = \mathcal{P}\eta\delta\mathcal{P}$. This implies that $[\eta][\delta] = \lambda[\eta\delta]$ for some $\lambda \in \mathbb{Z}$. Now, for a fixed complex-valued Haar measure d on G , one defines the augmentation map

$$\epsilon : \mathcal{H}_{\mathbb{Z}}(G, \mathcal{P}, 1) \rightarrow \mathbb{C} \quad f \mapsto \int_G f(t) dt$$

and checks that it respects the multiplication. (This follows from Proposition 4.2 of [3], if one takes for V the trivial representation and if one uses the embedding $\mathcal{H}_{\mathbb{Z}}(G, \mathcal{P}, 1) \subset \mathcal{H}_{\mathbb{C}}(G, \mathcal{P}, 1)$.) Therefore,

$$q^{l'(\eta)} q^{l'(\delta)} = \lambda q^{l'(\eta\delta)}.$$

This proves $\lambda = 1$. \square

This is sufficient to deduce the first part of the theorem:

Corollary 3.11. Let $[\eta]_f, [\delta]_g \in \mathcal{H}(G, \mathcal{P}, V)$ with $\epsilon, \delta \in W$ and such that $l(\eta\delta) = l(\eta) + l(\delta)$. Then

$$[\eta]_f * [\delta]_g = [\eta\delta]_{fg}.$$

Proof. Theorem 3.10 implies that there occurs only one admissible pair (k_1, k_2) when using formula (4) to calculate $[\eta] * [\delta]$ in $\mathcal{H}_{\mathbb{Z}}(G, \mathcal{P}, 1)$. It is obvious that the choice $k_1 = 1, k_2 = 1$ is admissible, hence this is the unique admissible pair. By the remark below the formula, this fact is still valid when calculating $[\eta]_f * [\delta]_g$, hence we get $h_{\eta\delta} = f \circ g$. \square

In order to establish the second part – as t^a is universally length-adding and central if $2|a$ – it suffices to evaluate

$$[\eta]_f * [\delta]_g$$

(where we will suppress the symbol $*$ from now on) for the 8 choices for $\eta, \delta \in \{w, w', tw = w't, t^{-1}w' = wt^{-1}\}$ where there is no addition of lengths. We first calculate

- $\mathcal{P}^{(w)} = \mathcal{P}^{(wt^{-1})} = \begin{pmatrix} \text{GL}_k(\mathcal{O}) & \mathbb{M}_{k \times k}(\mathfrak{P}) \\ \mathbb{M}_{k \times k}(\mathfrak{P}) & \text{GL}_k(\mathcal{O}) \end{pmatrix};$
- $\mathcal{P}^{(w')} = \mathcal{P}^{(w't)} = \begin{pmatrix} \text{GL}_k(\mathcal{O}) & \mathbb{M}_{k \times k}(\mathcal{O}) \\ \mathbb{M}_{k \times k}(\mathfrak{P}^2) & \text{GL}_k(\mathcal{O}) \end{pmatrix}.$

Therefore, in the first two cases we can take $\{\alpha_x = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \mid x \in \mathbb{M}_{k \times k}(\mathbb{F}_q)\}$ and in the last two cases we can take $\{\beta_y = \begin{pmatrix} 1 & \\ & \varpi y \end{pmatrix} \mid y \in \mathbb{M}_{k \times k}(\mathbb{F}_q)\}$ as set of representatives for $\mathcal{P}^{(\eta)} \setminus \mathcal{P}$.

Proposition 3.12.

$$\mathcal{P}w\mathcal{P}w\mathcal{P} \subset \left(\mathcal{P} \cup \mathcal{P}w\mathcal{P} \cup \bigcup_{x \in X} \mathcal{P}x\mathcal{P} \right),$$

where X can be taken as a subset of $\tilde{\mathcal{W}} - W$.

Proof. This follows easily from the observation

$$\mathcal{P}w\mathcal{P}w\mathcal{P} \subset \text{GL}_n(\mathcal{O}) = \bigcup_{x \in S_n} \mathcal{I}x\mathcal{I} = \bigcup_{x \in \tilde{\mathcal{W}} \cap S_n} \mathcal{P}x\mathcal{P}$$

and the decomposition

$$\tilde{\mathcal{W}} \cap S_n = ((\tilde{\mathcal{W}} - W) \cap S_n) \sqcup (W \cap S_n) = ((\tilde{\mathcal{W}} - W) \cap S_n) \sqcup \{1, w\}.$$

\square

Theorem 3.13 (Case 1: $\eta = \delta = w$). *In $\mathcal{H}_R(G, \mathcal{P}, V)$, we have the identity*

$$[w]_f[w]_g = \tau[1]_{fg} + [w]_{fg}^1.$$

Proof. By the above proposition, we know that $[w]_f[w]_g = [1]_X + [w]_Y$, so we just have to work out X and Y . For the first summand, we first remark that by [9], I.3.4.(iv), in $\mathcal{H}_{\mathbb{Z}}(G, P, 1)$, we have $[w][w] = \lambda[1] + \dots$, where $\lambda = \#\mathcal{P}w\mathcal{P}/\mathcal{P} = \#\mathcal{P}/\mathcal{P}^{(w)} = \#\mathcal{P}^{(w)} \setminus \mathcal{P} = q^{(k^2)}$. This means, we have to locate $q^{(k^2)}$ admissible pairs in formula (4). This is easily achieved by fixing $k_1 = 1$ and letting k_2 run through all $q^{(k^2)}$ possible values in $\mathcal{P}^{(w)} \setminus \mathcal{P}$. Applied to $\mathcal{H}_R(G, \mathcal{P}, V)$, this means we get $X = q^{(k^2)}f \circ g = \tau fg$.

For the second factor, the condition of the formula is $k_0 = wk_{x_1}^{-1}wk_{x_2}^{-1}w \in \mathcal{P}$, which boils down to

$$\begin{pmatrix} x_2 & 1 \\ 1+x_1x_2 & x_1 \end{pmatrix} \in \mathcal{P}.$$

This is possible if and only if x_1, x_2 are invertible and fulfill $x_1 x_2 = -1$. This gives $\# \text{GL}_k(q)$ -many admissible pairs and rise to

$$Y = \sum_{z \in \text{GL}_k(q)} \rho \left(\begin{smallmatrix} z & 1 \\ & -z^{-1} \end{smallmatrix} \right) \circ f \circ g = T^* f g.$$

□

Proposition 3.14.

$$\mathcal{P} t w \mathcal{P} w t^{-1} \mathcal{P} \subset \mathcal{P} \cup \mathcal{P} w' \mathcal{P} \cup \bigcup_{x \in Y} \mathcal{P} x \mathcal{P},$$

where Y can be taken as a subset of $\tilde{\mathcal{W}} - W$.

Proof. We use that $l(tw) = l(t) + l(w)$ to write

$$\begin{aligned} \mathcal{P} t w \mathcal{P} w t^{-1} \mathcal{P} &= \mathcal{P} t \mathcal{P} w \mathcal{P} w \mathcal{P} t^{-1} \mathcal{P} \subset \mathcal{P} t \mathcal{P} t^{-1} \mathcal{P} \cup \mathcal{P} t \mathcal{P} w \mathcal{P} t^{-1} \mathcal{P} \cup \bigcup_{x \in X} \mathcal{P} t \mathcal{P} x \mathcal{P} t^{-1} \mathcal{P} \\ &= \mathcal{P} \cup \mathcal{P} w' \mathcal{P} \cup \bigcup_{x \in X} \mathcal{P} t x t^{-1} \mathcal{P}. \end{aligned}$$

It is clear that conjugation by t preserves the property of being a member of W or not. □

Theorem 3.15. In $\mathcal{H}_R(G, \mathcal{P}, V)$ we have the identity

$$[tw]_1 [wt^{-1}]_1 = \tau[1]_1 + [w']^1.$$

Proof. By the proposition and the same reasoning as above, we see that $[tw]_1 [wt^{-1}]_1 = [1]_X + [w']_Y$, and that we have to locate $q^{(k^2)}$ admissible pairs in formula (4) in order to calculate X , which can again achieved by fixing $k_1 = 1$, yielding $X = q^{(k^2)}$. For the second summand, we can take $k_1 = \alpha_x$ and $k_2 = \beta_y$. We have to work out for which values of x, y we have

$$t w t^{-1} k_2^{-1} t w k_1^{-1} w t^{-1} \in \mathcal{P}.$$

This boils down to

$$\left(\begin{smallmatrix} y & \varpi^{-1}(\mathbf{1} + xy) \\ \varpi & x \end{smallmatrix} \right) \in \mathcal{P}.$$

This is clearly fulfilled if and only if $x, y \in \text{GL}_k(q)$ and fulfill $xy = -1$. This gives the second summand as $[tw t^{-1}]_{T^*} = [w']^1$. □

Corollary 3.16 (Case 2: $\eta = tw, \delta = tw'$).

$$[tw]_f [wt^{-1}]_g = \tau[1]_{fg} + [tw t^{-1}]_{fg}^1.$$

Proof. Write

$$[tw]_f [wt^{-1}]_g = [1]_f [tw]_1 [wt^{-1}]_1 [1]_g$$

and apply the theorem above. □

Corollary 3.17 (Case 3: $\eta = \delta = w'$).

$$[w']_f [w']_g = \tau[1]_{fg} + [w']_{fg}^1.$$

Proof. Write

$$[w']_f[w']_g = [tw]_1[t^{-1}]_f[t]_g[wt^{-1}]_1 = [tw]_1[1]_{fg}[wt^{-1}]_1 = [tw]_1[wt^{-1}]_1[1]_{fg}$$

and apply the theorem above. \square

Proposition 3.18.

$$\mathcal{P}wt\mathcal{P}t^{-1}w\mathcal{P} \subset \mathcal{P} \cup \mathcal{P}w\mathcal{P} \cup \bigcup_{x \in Y} \mathcal{P}x\mathcal{P},$$

where Y can be taken as a subset of $\tilde{\mathcal{W}} - W$.

Proof. This becomes clear when writing

$$\mathcal{P}wt\mathcal{P}t^{-1}w\mathcal{P} = \mathcal{P}w\mathcal{P}t\mathcal{P}t^{-1}\mathcal{P}w\mathcal{P} = \mathcal{P}w\mathcal{P}w\mathcal{P}.$$

\square

Theorem 3.19.

$$[wt^{-1}]_1[tw]_1 = \tau[1]_1 + [w]^1.$$

Proof. Analogous to the theorem above. \square

Corollary 3.20 (Case 4: $\eta = tw', \delta = tw$).

$$[wt^{-1}]_f[tw]_g = \tau[1]_{fg} + [w]_{fg}^1.$$

Theorem 3.21 (Case 5: $\eta = w, \delta = wt^{-1}$).

$$[w]_f[wt^{-1}]_g = \tau[t^{-1}]_{fg} + [wt^{-1}]_{fg}^1.$$

Proof. The reasoning is now standard: We first check that

$$\mathcal{P}w\mathcal{P}wt^{-1}\mathcal{P} = \mathcal{P}w\mathcal{P}w\mathcal{P}t^{-1}\mathcal{P} \subset \mathcal{P}t^{-1}\mathcal{P} \cup \mathcal{P}wt^{-1}\mathcal{P} \cup \bigcup_{x \in Y} \mathcal{P}x\mathcal{P},$$

where $Y \cap W = \emptyset$ (this follows from the observation that multiplication by t^{-1} does respect the property of being contained in W or not). Now, in $\mathcal{H}_{\mathbb{Z}}(G, \mathcal{P}, 1)$, we have $[w][wt^{-1}] = \lambda[t^{-1}] + \dots$ with $\lambda = \#(\mathcal{P}w\mathcal{P} \cap t^{-1}\mathcal{P}tw\mathcal{P}) \setminus \mathcal{P}$. We use that \mathcal{P} is stable under conjugation with t , so $\lambda = q^{(n^2)}$, and we can use the same reasoning as above to establish the first summand as $[t^{-1}]_{\tau fg}^1$.

For the second summand, we start with $k_1 = \alpha_x, k_2 = \beta_y$ and have to figure out when $wt^{-1}k_2^{-1}twk_1^{-1}w \in \mathcal{P}$. This boils down to the condition

$$\begin{pmatrix} x & 1 \\ 1+xy & y \end{pmatrix} \in \mathcal{P}.$$

We get $[wt^{-1}]_{fg}^1$. \square

Corollary 3.22 (Case 6: $\eta = t^{-1}w', \delta = w'$).

$$[t^{-1}w']_f[w']_g = \tau[t^{-1}]_{fg} + [t^{-1}w']_{fg}^1.$$

Proof. This follows from the above when writing

$$[t^{-1}w']_f[w']_g = [wt^{-1}]_f[t]_g[wt^{-1}]_1 = [w]_{fg}[wt^{-1}]_1.$$

□

The following is proved completely analogously:

Theorem 3.23 (Case 7: $\eta = tw, \delta = w$).

$$[tw]_f[w]_g = \tau[t]_{fg} + [tw]_{fg}^1.$$

Corollary 3.24 (Case 8: $\eta = w', \delta = w't$).

$$[w']_f[w't]_g = \tau[t]_{fg} + [w't]_{fg}^1.$$

4 Tensor decomposition

4.1 Twisted tensor products

For our decomposition of the Hecke algebra we need a mild generalization of the twisted tensor product. This is a construction originating from the theory of quantum groups and treated e. g. in [5].

For this subsection, let A, B be associative R -algebras over some commutative unital ring R . Denote the multiplication maps by $\mu_A : A \times A \rightarrow A$ and $\mu_B : B \times B \rightarrow B$. Let S be an associative unital R -algebra together with R -algebra homomorphisms $S \rightarrow A, S \rightarrow B$. This allows us to regard A and B as S - S -bimodules in a way which is compatible with their structures as R -algebras.

Definition 4.1 (Twisting map). A twisting map (over S) is an S -linear map

$$\psi : B \otimes_S A \rightarrow A \otimes_S B$$

which fulfills $\psi(1_B \otimes a) = a \otimes 1_B$ and $\psi(b \otimes 1_A) = 1_A \otimes b$.

If ψ is a twisting map, the multiplication $\mu_\psi := (\mu_A \otimes \mu_B) \circ (\text{id}_A \otimes \psi \otimes \text{id}_B)$ makes $A \otimes_S B$ into an R -algebra and S - S -bimodule denoted by $A \overset{\psi}{\otimes}_S B$. We can write the product explicitly as

$$(a_1 \otimes b_1) * (a_2 \otimes b_2) = \sum_i (a_1 *_A a_2^i) \otimes (b_1^i *_B b_2)$$

if $\psi(b_1 \otimes a_2) = \sum_i a_2^i \otimes b_1^i$.

Remark 4.2. The question whether $A \overset{\psi}{\otimes}_S B$ is associative can easily be translated into a property of the map ψ , cf. sections 2.3-2.4 of [5]. In our applications we will be constructing R -algebra isomorphisms

$$A \overset{\psi}{\otimes}_S B \xrightarrow{\cong} C$$

where C is known to be associative. This is why we do not have to check associativity separately for the twisted tensor products we are about to consider.

Proposition 4.3. *Let R, S, A, B as above and consider twisting maps $\psi_1 : S \otimes_R A \rightarrow A \otimes_R S$ (over R) and $\psi_2 : B \otimes_S (A \otimes_R^{\psi_1} S) \rightarrow (A \otimes_R^{\psi_1} S) \otimes_S B$ (over S). Then there is an isomorphism of R -algebras*

$$A \otimes_R^{\psi_3} B \cong (A \otimes_R^{\psi_1} S) \otimes_S^{\psi_2} B, \quad (5)$$

where $\psi_3 = \gamma \circ \psi_2 \circ (\text{id}_B \otimes \iota_A)$. Here, we made use of the maps

$$\iota_A : A \rightarrow A \otimes_R^{\psi_1} S \quad a \mapsto a \otimes 1$$

and

$$\gamma : (A \otimes_R^{\psi_1} S) \otimes_S B \rightarrow A \otimes_R B \quad \sum_i a_i \otimes s_i \otimes b_i \mapsto \sum_i a_i \otimes s_i \cdot b_i.$$

Proof. Clearly, ψ_3 defines a twisting map over R . Moreover it is clear that the assignment $f : \sum_i a_i \otimes b_i \mapsto \sum_i a_i \otimes 1 \otimes b_i$ establishes the isomorphism (5) on the level of R -modules. Thus we have to check that f commutes with the algebra multiplication, which boils down to the formal calculation

$$(f \times f) \circ \mu_{\psi_3} = \mu_{\psi_2} \circ (f \times f). \quad \square$$

4.2 Decomposing $\mathcal{H}(G, \mathcal{P}, V)$

Definition 4.4. For $1 \leq \tau < \ell$, denote by $R[T]^\tau$ the R -algebra with underlying R -module $R[T]$, but modified multiplication defined by

$$T^a * T^b = \begin{cases} \tau T^{a+b} + T^{a+b+1} & \text{if both } a \text{ and } b \text{ are odd;} \\ T^{a+b} & \text{otherwise.} \end{cases}$$

Lemma 4.5. $R[T]^\tau$ is Euclidean (and, hence, a principal ideal domain).

Proof. First, consider the ring homomorphism

$$E : R[X, Y] \rightarrow R[T] \quad X \mapsto T, Y \mapsto T^2.$$

E is clearly surjective, and we want to show that $\ker(E) = (f_0)$ with $f_0 = X^2 - Y(X + \tau)$. The inclusion “ \supset ” is clear, so let f be an element of the kernel. As $E(f_0) = 0$, we can successively replace the symbol X^2 in the expression of f by $Y(X + \tau)$, and via this process we will end up with a polynomial of the form $f' = \sum_i a_i Y^i + X \sum_i b_i Y^i$ such that $E(f) = E(f')$. But we have

$$E(f') = \sum_i (a_i T^{2i} + b_i T^{2i+1}),$$

so if this expression vanishes we must have $f' = 0$. We conclude that any $f \in \ker(E)$ can be transformed into the zero polynomial by replacing X^2 by $Y(X + \tau)$, i. e. $f \in (f_0)$. Hence

$$R[X, Y]/(f_0) \cong R[T]^\tau.$$

Now, we can embedd

$$R[X, Y]/(f_0) \hookrightarrow R(X) \quad X \mapsto X, Y \mapsto \frac{X^2}{X + \tau}.$$

In order to determine the image of this embedding, we write $X^2/(X + \tau) = X - \tau + \tau(X + \tau)^{-1}$, so we get an isomorphism $R[X, Y]/(f_0) \cong R[X]_{(X + \tau)}$ (localization at $X + \tau$). As the property of being Euclidean is preserved by localization, the statement follows. \square

Definition 4.6. Denote by $\mathcal{H}_R^\dagger(G, \mathcal{P}, V)$ the subalgebra of $\mathcal{H}_R(G, \mathcal{P}, V)$ spanned by all elements of the form $[\eta]^a$ with $a \in \mathbb{N}_0$ and $\eta \in W$ (where, in order to make sense, a must be even (resp. odd) if η is diagonal (resp. non-diagonal)).

Let $\mathcal{K} = \mathrm{GL}_n(\mathcal{O})$ denote the maximal open compact subgroup of G .

Definition 4.7. By $\mathcal{H}_R^\dagger(\mathcal{K}, \mathcal{P}, V)$ we denote the intersection $\mathcal{H}_R^\dagger(G, \mathcal{P}, V) \cap \mathcal{H}_R(\mathcal{K}, \mathcal{P}, V)$. For \mathcal{P} the standard parabolic subgroup of \mathcal{G} with Levi component \mathcal{M} , define the Hecke algebra $\mathcal{H}_R(\mathcal{G}, \mathcal{P}, V)$ consisting of maps $\mathcal{G} \rightarrow V$ fulfilling a bi-invariance property analogous to (3) (see also section 5.1). Then, because of the obvious isomorphism

$$\mathcal{H}_R(\mathcal{K}, \mathcal{P}, V) \cong \mathcal{H}_R(\mathcal{G}, \mathcal{P}, V),$$

we can equally well define $\mathcal{H}_R^\dagger(\mathcal{K}, \mathcal{P}, V)$ as the subalgebra of $\mathcal{H}_R(\mathcal{G}, \mathcal{P}, V)$ spanned by all elements of the form $[\eta]^a$ with $\eta \in \{1, w\}, a \in \mathbb{N}_0$. This motivates the notation

$$\mathcal{H}_R^\dagger(\mathcal{G}, \mathcal{P}, V) = \mathcal{H}_R^\dagger(\mathcal{K}, \mathcal{P}, V).$$

Definition 4.8 (Characteristic polynomial). It is clear that the assignment $\sum_I r_i T^i \mapsto \sum_I r_i [w^i]^i$ defines an R -algebra homomorphism

$$R[T]^\tau \rightarrow \mathcal{H}_R(\mathcal{G}, \mathcal{P}, V) = \mathrm{End}_{\mathcal{G}}(\mathrm{ind}_{\mathcal{P}}^{\mathcal{G}}(V))$$

with image

$$\mathcal{H}_R^\dagger(\mathcal{G}, \mathcal{P}, V) = \langle [w^a]^a \mid a \in \mathbb{N}_0 \rangle_R.$$

We define $F = F_{R,q,k}$ to be the minimal-degree monic polynomial which generates the kernel of this surjection. It is clear that F depends only on the finite group data (R, q, k) .

Example 4.9. If ℓ divides $q-1$, we have $\tau = 1$ and we get $F = T^2$ (and, consequently, an isomorphism between $R[T]^\tau$ and the dual numbers $R[\epsilon]$) if $T^*.V \neq 0$ and $F = T$ (and $R[T]^\tau \cong R$) if $T^*.V = 0$.

It is a natural question to ask if and how these polynomials are related for different choices of the data (R, q, k) . In particular, one could conjecture that they depend only on R and q^k , i. e. that $F_{R,q,ab} = F_{R,q^a,b}$ for any choice of $a, b \geq 1$.

Lemma 4.10 (Failure of Commutativity). *In $\mathcal{H}_R(G, \mathcal{P}, V)$ we have $[1]_f[\eta]^a = [\eta]^a[1]_f$ and*

•

$$[w]_f[\eta]^a = [w\eta w]^a[w]_f \quad \text{if } \eta = t^{2b}w' \dots w \text{ or } \eta = t^{2b+1}w \dots w;$$

•

$$\begin{aligned} [w]_f[\eta]^a &= (\tau[w\eta w]^a + [\eta w]^{a+1})[w]_f \\ &= \tau[w\eta w]^a[w]_f + [\eta]^a[1]_f^1 \quad \text{if } \eta = t^{2b}w \dots w \text{ or } \eta = t^{2b+1}w' \dots w; \end{aligned}$$

•

$$\begin{aligned} [w]_f[\eta]^a &= \tau^{-1}([w\eta w]^a - [w\eta]^{a+1})[w]_f \\ &= \tau^{-1}[w\eta w]^a([w]_f - [1]_f^1) \quad \text{if } \eta = t^{2b}w' \dots w' \text{ or } \eta = t^{2b+1}w \dots w'; \end{aligned}$$

•

$$\begin{aligned} [w]_f[\eta]^a &= ([w\eta w]^a - [w\eta]^{a+1} + \tau^{-1}[\eta w]^{a+1} - \tau^{-1}[\eta]^{a+2})[w]_f \\ &= [w\eta w]^a[w]_f + ([\eta]^a - [w\eta w]^a)[1]_f^1 \quad \text{if } \eta = t^{2b}w \dots w' \text{ or } \eta = t^{2b+1}w' \dots w'. \end{aligned}$$

Proof. Straight-forward calculation. \square

Proposition 4.11. *A diagonal element $\delta_{x,y} := \text{diag}(\varpi^x, \dots, \varpi^x, \varpi^y, \dots, \varpi^y) \in W$ is of the form $t^{2b}w' \dots w$ or $t^{2b+1}w \dots w$ if and only if $x \leq y$ and of the form $t^{2b}w \dots w'$ or $t^{2b+1}w' \dots w'$ iff $x \geq y$.*

Proof. It is an easy calculation that

- $\delta_{x,y} = t^{y+x}(w'w)^{(y-x)/2}$ if $y \geq x$ and $y - x \equiv 0 \pmod{2}$;
- $\delta_{x,y} = t^{y+x}w(w'w)^{(y-x-1)/2}$ if $y \geq x$ and $y - x \equiv 1 \pmod{2}$;
- $\delta_{x,y} = t^{y+x}(ww')^{(x-y)/2}$ if $y \leq x$ and $y - x \equiv 0 \pmod{2}$;
- $\delta_{x,y} = t^{y+x}(w'w)^{(x-y-1)/2}w'$ if $y \leq x$ and $y - x \equiv 1 \pmod{2}$.

\square

Theorem 4.12. *The assignment*

$$\psi : R[T]_{/F}^\tau \otimes_R R[\mathbb{Z}^2] \rightarrow R[\mathbb{Z}^2] \otimes_R R[T]_{/F}^\tau$$

$$\left(\sum_i r_i T^i \right) \otimes (\alpha, \beta) \mapsto \begin{cases} (\alpha, \beta) \otimes (\sum_{2|i} r_i T^i) + (\beta, \alpha) \otimes (\sum_{2 \nmid i} r_i T^i) & \text{if } \alpha \leq \beta; \\ (\alpha, \beta) \otimes (\sum_{2|i} (r_i + r_{i-1}) T^i) + (\beta, \alpha) \otimes (\sum_{2 \nmid i} r_i (T^i - T^{i+1})) & \text{if } \alpha \geq \beta; \end{cases}$$

defines a twisting map. (In this formula, the symbol r_{-1} is to be interpreted as 0.) Moreover, there is an isomorphism of R -algebras

$$R[\mathbb{Z}^2] \otimes_R^\psi R[T]_{/F}^\tau \cong \mathcal{H}^\dagger(G, \mathcal{P}, V) \quad (6)$$

induced by the assignment

$$E : (\alpha, \beta) \otimes \left(\sum_i r_i T^i \right) \mapsto \delta_{\alpha, \beta} * \left(\sum_i r_i [w^i]^i \right).$$

Proof. It is clear that ψ defines a twisting map. It follows from the failure-of-commutativity lemma that E gives rise to an R -algebra map. Thus, the isomorphism is established as soon as we find a set-theoretic inverse mapping G of E . For this, consider

$$G : [\eta]^a \mapsto \begin{cases} (x, y) \otimes T^a & \text{if } \eta = \delta_{x,y}; \\ (x, y) \otimes T^a & \text{if } \eta = \delta_{x,y}w \text{ and } x \geq y; \\ (x, y) \otimes \tau^{-1}(T^a - T^{a+1}) & \text{if } \eta = \delta_{x,y}w \text{ and } x < y. \end{cases}$$

It is now straightforward to check that $E \circ G = 1$ and $G \circ E = 1$. \square

Corollary 4.13. *The isomorphism (6) is an isomorphism of $R[T]_{/F}^\tau$ -bimodules via*

$$\begin{aligned} & \left(\sum_i r_i T^i \right) * ((\alpha, \beta) \otimes h) * g \\ &= \begin{cases} (\alpha, \beta) \otimes (\sum_{2|i} r_i T^i) h g + (\beta, \alpha) \otimes (\sum_{2 \nmid i} r_i T^i) h g & \text{if } \alpha \leq \beta; \\ (\alpha, \beta) \otimes (\sum_{2|i} (r_i + r_{i-1}) T^i) h g + (\beta, \alpha) \otimes (\sum_{2 \nmid i} r_i (T^i - T^{i+1})) h g & \text{if } \alpha \geq \beta; \end{cases} \end{aligned}$$

where $\sum_i r_i T^i, h, g \in R[T]_{/F}^\tau$ and $(\alpha, \beta) \in R[\mathbb{Z}^2]$.

Corollary 4.14.

$$\mathcal{H}^\dagger(\mathcal{G}, \mathcal{P}, V) \cong R[T]_{/F}^\tau.$$

The main decomposition is now

Theorem 4.15. *The assignment*

$$\begin{aligned} & [1]_f \otimes [\eta]^a \mapsto [\eta]^a \otimes [1]_f; \\ & [w]_f \otimes [\eta]^a \mapsto \begin{cases} [w\eta w]^a \otimes [w]_f & \text{if } \eta = t^{2b} w' \dots w \text{ or } t^{2b+1} w \dots w; \\ \tau[w\eta w]^a \otimes [w]_f + [\eta]^a \otimes [1]_f^1 & \text{if } \eta = t^{2b} w \dots w \text{ or } t^{2b+1} w' \dots w; \\ \tau^{-1}[w\eta w]^a \otimes ([w]_f - [1]_f^1) & \text{if } \eta = t^{2b} w' \dots w' \text{ or } t^{2b+1} w \dots w'; \\ [w\eta w]^a \otimes ([w]_f - [1]_f^1) + [\eta]^a \otimes [1]_f^1 & \text{if } \eta = t^{2b} w \dots w' \text{ or } t^{2b+1} w' \dots w'; \end{cases} \end{aligned}$$

defines a twisting map

$$\Psi : \mathcal{H}_R(\mathcal{G}, \mathcal{P}, V) \otimes_{\mathcal{H}_R^\dagger(\mathcal{G}, \mathcal{P}, V)} \mathcal{H}_R^\dagger(G, \mathcal{P}, V) \rightarrow \mathcal{H}_R^\dagger(G, \mathcal{P}, V) \otimes_{\mathcal{H}_R^\dagger(\mathcal{G}, \mathcal{P}, V)} \mathcal{H}_R(\mathcal{G}, \mathcal{P}, V).$$

This gives rise to an isomorphism of $\mathcal{H}_R^\dagger(\mathcal{G}, \mathcal{P}, V)$ - (and, hence, R -) algebras

$$\mathcal{H}_R^\dagger(G, \mathcal{P}, V) \otimes_{\mathcal{H}_R^\dagger(\mathcal{G}, \mathcal{P}, V)}^\Psi \mathcal{H}_R(\mathcal{G}, \mathcal{P}, V) \cong \mathcal{H}_R(G, \mathcal{P}, V).$$

Proof. Because of the obvious R -linearity of the given assignment, the verification that Ψ is well-defined and linear can be done case-by-case (depending on the structure of η). So, let η be as in the first case above, i. e. $\eta = t^{2b} w' \dots w$ or $t^{2b+1} w \dots w$. It suffices to compare the following expressions:

$$\Psi([1]_f [w]^b \otimes [\eta]^a) \stackrel{?}{=} \Psi([1]_f \otimes [w]^b [\eta]^a) \quad \text{and} \quad \Psi([w]_f [w]^b \otimes [\eta]^a) \stackrel{?}{=} \Psi([w]_f \otimes [w]^b [\eta]^a).$$

After evaluation, the first comparison is between $[w\eta w]^a \otimes [w]_f^b$ and $[w\eta]^{a+b} \otimes [1]_f$, which coincides as we have $[w\eta]^{a+b} = [w\eta w]^a [w]^b$. The second comparison, after evaluation, is between $\tau[\eta]^a \otimes [1]_f^b + [w\eta w]^a \otimes [w]_f^{b+1}$ and $\tau[\eta w]^{a+b} \otimes [w]_f + [w\eta]^{a+b} \otimes [1]_f^1$. Equality follows from the identities

$$\tau[1]_f^1 = ([w]^1 - [1]^2)[w]_f \quad \text{and} \quad [\eta]^c([w]^1 - [1]^2) = \tau[\eta w]^{c+1}. \quad (7)$$

For the $\mathcal{H}_R^\dagger(\mathcal{G}, \mathcal{B}, V)$ -linearity, it is clearly sufficient to prove that $[w]^1 \circ \Psi = \Psi \circ [w]^1$. We have

$$[w]^1 \circ \Psi([1]_f \otimes [\eta]^a) = [w]^1 ([\eta]^a \otimes [1]_f) = [w\eta]^{a+1} \otimes [1]_f.$$

On the other hand,

$$\Psi \circ [w]^1([1]_f \otimes [\eta]^a) = \Psi([w]_f^1 \otimes [\eta]^a) = \Psi([1]_f \otimes [w\eta]^{a+1}) = [w\eta]^{a+1} \otimes [1]_f.$$

Moreover, we have

$$[w]^1 \circ \Psi([w]_f \otimes [\eta]^a) = [w]^1([w\eta w]^a \otimes [w]_f) = (\tau[w\eta]^{a+1} + [w\eta w]^{a+2}) \otimes [w]_f.$$

On the other hand,

$$\Psi \circ [w]^1([w]_f \otimes [\eta]^a) = \Psi((\tau[1]_f^1 + [w]_f^2) \otimes [\eta]^a) = \tau[\eta]^a \otimes [1]_f^1 + [w\eta w]^a \otimes [w]_f^2.$$

Here, we used the identity in (7) again. The other three cases are checked similarly.

That $\Psi(1 \otimes X) = X \otimes 1$ and $\Psi(Y \otimes 1) = 1 \otimes Y$ is obvious. (Remark that $\eta = 1$ is included in both the first and the last case in the statement of the theorem.)

Now the desired isomorphism is given by

$$E : [\eta]^a \otimes ([1]_f + [w]_g) \mapsto [\eta]^a ([1]_f + [w]_g).$$

This defines an R -algebra homomorphism by the failure-of-commutativity lemma. In order to see that it is bijective it is sufficient to give a set-theoretic inverse map. Similar to the last proof, we consider

$$G : [\eta]_f \mapsto \begin{cases} [\eta]^0 \otimes [1]_f & \text{if } \eta \text{ diagonal;} \\ [\eta w]^0 \otimes [w]_f & \text{if } \eta \text{ non-diagonal, ends on } w; \\ \tau^{-1}[\eta w]^0 \otimes ([w]_f - [1]_f^1) & \text{if } \eta \text{ non-diagonal, does not end on } w. \end{cases}$$

(The third case includes $\eta = t^{2b+1}$.) It is clear that $E \circ G = 1$. To check that $G \circ E = 1$ on symbols of the form $[\eta]^a \otimes [1]_f$ involves checking three cases (depending on the form of η , as distinguished in the definition of G), which is straightforward. To check that $G \circ E = 1$ on symbols of the form $[\eta]^a \otimes [w]_f$ boils down to checking four cases, which, again, poses no difficulty. \square

A variation of this decomposition is:

Corollary 4.16. *The assignment*

$$\begin{aligned} [1]_f \otimes (\alpha, \beta) &\mapsto (\alpha, \beta) \otimes [1]_f; \\ [w]_f \otimes (\alpha, \beta) &\mapsto \begin{cases} (\beta, \alpha) \otimes [w]_f & \text{if } \alpha \leq \beta; \\ (\alpha, \beta) \otimes [1]_f^1 + (\beta, \alpha) \otimes ([w]_f - [1]_f^1) & \text{if } \alpha \geq \beta; \end{cases} \end{aligned}$$

defines a twisting map

$$\zeta : \mathcal{H}_R(\mathcal{G}, \mathcal{P}, V) \otimes_R R[\mathbb{Z}^2] \rightarrow R[\mathbb{Z}^2] \otimes_R \mathcal{H}_R(\mathcal{G}, \mathcal{P}, V).$$

This gives rise to an isomorphism of R -algebras

$$R[\mathbb{Z}^2] \otimes_R^\zeta \mathcal{H}_R(\mathcal{G}, \mathcal{P}, V) \cong \mathcal{H}_R(G, P, V).$$

Proof. By the results above, we know that

$$\mathcal{H}(G, P, V) = \left(R[\mathbb{Z}^2] \otimes_R^\psi R[T]_{/F}^\tau \right) \otimes_{R[T]_{/F}^\tau}^\Psi \mathcal{H}_R(\mathcal{G}, \mathcal{P}, V).$$

Therefore, the claim follows from Proposition 4.3. \square

Example 4.17 (Scalar Iwahori-Hecke algebra). We can recover the scalar case (i. e. where $G = \mathrm{GL}_2, k = 1$ and V is the trivial representations) as follows: Let $W = S_2$ be the Weyl group of G , then it is easy to see that $\mathcal{H}(\mathcal{G}, \mathcal{B}, V)$ is isomorphic to the combinatorial Hecke algebra $\mathcal{H}(W, q)$ (cf. [9], I.3.13, where this algebra is referred to as $H_R^0(2, q)$). As we are interested in the modular case $\ell \nmid \# \mathrm{GL}_2(q)$, we have either $\ell \mid (q - 1)$ or $\ell \mid (q + 1)$. Let us assume that we are in the first case, so that $\mathcal{H}(W, q)$ is isomorphic the group algebra of W . If we assume moreover that $q > 2$, we see that $T^*|V = 0$ and hence the map ζ from the above corollary comes from the action

$$\xi : S_2 \rightarrow \mathrm{Aut}(\mathbb{Z}^2) \quad 1.(x, y) = (x, y) \text{ and } w.(x, y) = (y, x).$$

Let $I \subset G$ denote the Iwahori subgroup and \tilde{W} the affine Weyl group of G . We conclude that

$$\mathcal{H}(G, I, 1) \cong R[\mathbb{Z}^2] \overset{\zeta}{\otimes}_R R[S_2] \cong R[\mathbb{Z}^2 \rtimes_\xi S_2] = R[\tilde{W}],$$

what matches with the classical description of $\mathcal{H}(G, I, 1)$ ([9], I.3.14). In the case $\ell \mid q + 1$ one gets correspondingly an isomorphism between $\mathcal{H}(G, I, 1)$ and the combinatorial (affine) Hecke algebra $H(\tilde{W}, -1)$.

5 Relating simple blocks of different groups

Retain the assumptions on F and R from the introduction and fix two numbers $k, m \geq 1$ and denote $G_1 = \mathrm{GL}_{2km}(F), G_2 = \mathrm{GL}_{2k}(F^m)$, where F^m denotes the unramified extension of F of degree m . Moreover, let $P_i \subset G_i$ be the standard parabolic subgroups characterized by having a Levi decomposition

$$P_i = M_i U_i \text{ with } M_i \cong H_i \times H_i,$$

where $H_1 = \mathrm{GL}_{km}(F)$ and $H_2 = \mathrm{GL}_k(F^m)$.

Consider two irreducible supercuspidal level-0 representations $\pi_i \in \mathrm{Rep}_R(H_i)$ (for $i = 1, 2$) and denote $\pi_i^2 = \pi_i \boxtimes \pi_i \in \mathrm{Rep}_R(M_i)$. Having the banal situation in mind, it is natural to consider the following conjecture:

Conjecture 5.1. There is an equivalence of categories

$$\mathfrak{R}_R^{[M_1, \pi_1^2]_{G_1}}(G_1) \cong \mathfrak{R}_R^{[M_2, \pi_2^2]_{G_2}}(G_2).$$

Remark 5.2. The case where $k = 1$ is the most interesting and also the most general, as the proof for an arbitrary k can be given by applying the conjecture two times in the $k = 1$ -version.

5.1 Connection with a conjecture on finite groups

For this final subsection (where we restrict to the $k = 1$ case as suggested in the above remark), consider the following (self-contained but slightly differing) notation:

- R denotes an algebraically closed field of positive characteristic ℓ ;
- q denotes a power of a prime different from ℓ ;
- $m \in \mathbb{N}_+$ and $\mathcal{G}_1 = \mathrm{GL}_{2m}(q), \mathcal{H}_1 = \mathrm{GL}_m(q), \mathcal{G}_2 = \mathrm{GL}_2(q^m), \mathcal{H}_2 = \mathrm{GL}_1(q^m) = \mathbb{F}_{q^m}^\times$;

- $w_1 \in \mathcal{G}_1$ denotes the permutation matrix corresponding to $(1, m+1)(2, m+2) \dots (m, 2m) \in S_{2m}$ and $w_2 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \in \mathcal{G}_2$ denotes the permutation matrix corresponding to $(1, 2) \in S_2$;
- \mathcal{P}_i denotes the standard parabolic subgroup of \mathcal{G}_i characterized by admitting a Levi decomposition with Levi factor $\mathcal{M}_i = \mathcal{H}_i \times \mathcal{H}_i$;
- \mathbf{i}_i denotes the Harish-Chandra induction functor from representations of \mathcal{M}_i to representations of \mathcal{G}_i ;
- $\pi_2 = R$ denotes the trivial character of \mathcal{H}_2 and π_1 denotes some supercuspidal irreducible representation of \mathcal{H}_1 over R . π_i^2 denotes $\pi_i \boxtimes \pi_i$ as a representation of \mathcal{M}_i .

For V_i a representation of \mathcal{M}_i , the elements of the Hecke algebra $\mathcal{H}_R(\mathcal{G}_i, \mathcal{M}_i, V_i) = \text{End}_{\mathcal{G}_i}(\mathbf{i}_i(V_i))$ can be characterized (by [9], I.8.5) as maps $\varphi : G \rightarrow \text{End}_R(V_i)$ with a similar bi-equivariance condition as in (3). Thus, denote by $T_{V_i}^*$ the element of $\mathcal{H}_R(\mathcal{G}_i, \mathcal{M}_i, V_i)$ which is supported on $\mathcal{P}_i w_i \mathcal{P}_i$ and determined by

$$w_i \xrightarrow{T_{V_i}^*} \sum_{g \in \mathcal{H}_i} \begin{pmatrix} g & \\ & -g^{-1} \end{pmatrix}.$$

Conjecture 5.3. There exist representations $V_i \in \text{Rep}_R(\mathcal{M}_i)$ ($i = 1, 2$) fulfilling

- $V_i^{\text{ss}} \cong \oplus \pi_i^2$ (where ss denotes the semi-simplification), i. e. all JH-constituents of V_i are isomorphic to π_i^2 ;
- V_i is projective and isomorphic to its contragredient;

such that there exists an isomorphism of R -algebras

$$\varphi : \mathcal{H}_R(\mathcal{G}_1, \mathcal{M}_1, V_1) \rightarrow \mathcal{H}_R(\mathcal{G}_2, \mathcal{M}_2, V_2)$$

which maps $T_{V_1}^*$ to $T_{V_2}^*$.

If Conjecture 5.3 holds true, it seems reasonable to expect that the self-dual pro-generators constructed in Theorem 2.2 will do the job. However, independent of the origin of these V_i and as outlined in the introduction, we can make use of the results of the previous chapter (in particular of Corollary 4.16) to deduce the following:

Corollary 5.4. *Conjecture 5.3 implies Conjecture 5.1. Moreover, it implies that the conjectured identity stated below Example 4.9 holds.*

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